# Applications of Pology to Stripping Reactions*

C. DULLEMOND†
*Department of Physics and Astronomy, University of Rochester, Rochester, New York*

AND

H. J. SCHNITZER††
*Department of Physics and Astronomy, University of Rochester, Rochester, New York and Department of Physics, Brandeis University, Waltham, Massachusetts*

(Received 24 August 1962)

A model-independent definition for a reduced width in nuclear reactions is proposed. In particular, stripping reactions of the form $A(d,p)B$ are studied. The applicability of pology techniques for the determination of absolute reduced widths is tested by means of a computer experiment. The results indicate that the extrapolation to the pole can give a model-independent value for the reduced width which is accurate to at least an order of magnitude. It is likely that increased accuracy is possible if such model-dependent features as the bound-state wave function are included in the extrapolation formula.

## I. INTRODUCTION

THERE exists a similarity between renormalized coupling constants in elementary particle interactions and reduced widths in nuclear interactions. In a process like $A+B \rightleftarrows C$ where $A$, $B$, and $C$ are elementary particles, the renormalized coupling constant is a measure of the strength of such an interaction; if $A$, $B$, and $C$ are complex nuclei, the strength is indicated by the reduced width. For elementary particles, the restriction that $A$, $B$, and $C$ simultaneously be on their mass shells defines the coupling constant; similarly, a reduced width alone will be enough to measure the strength of such nuclear interactions if the nuclei are on their mass shells. It appears, therefore, to be useful to define a reduced width in a model-independent way, since a coupling constant ought to be independent of the details of the interaction at short distances. In particular, the nuclear radius should not appear as a parameter. One can carry the analogy further and propose that the scattering amplitude for nuclear reactions is an analytic function in the complex $\cos\theta$ plane, with singularities as suggested by Feynman graphs.[1,2] As has been shown by Chew and Low,[3] it is possible to regard the virtual process $A+B \rightarrow C$ as part of a real process with a one-particle intermediate state (either $A$, $B$, or $C$); then the renormalized coupling constant appears as a factor in the residue of a pole in the scattering amplitude. It often happens, that this pole lies close to the physical region. Then, when the influence of singularities in the neighborhood of the pole is small enough so that the pole term still dominates the scattering amplitude in parts of the physical region, the residue of the pole can be found by means of extrapolation in a model-independent way. Let $\sigma_{\text{exp}}$ be the experimental value of the differential cross section for the stripping reaction $A(d,p)B$ as a function of the c.m. scattering angle. Let $\sigma_{\text{pole}}$ be its theoretical value in the absence of all singularities, except the stripping pole. If one wants to know the reduced width for the reaction $A+n \rightarrow B$, then extrapolation of $\sigma_{\text{exp}}/\sigma_{\text{pole}}$ to the stripping pole should be sufficient. Of course, to do this successfully, $\sigma_{\text{exp}}$ should be quite accurately known for many values of the scattering angle.

Amado has proposed a Chew-Low extrapolation for stripping reactions[4] which extrapolates $\sigma_{\text{exp}}/\sigma_{\text{theor}}$, where $\sigma_{\text{theor}}$ depends on the Butler formula. It is the purpose of this paper to present a model-independent extrapolation as indicated above. We have chosen the stripping reaction $Si^{28}(d,p)Si^{29}$ for this purpose and obtain redefined reduced widths. We have used Satchler's[5] distorted-wave calculations of the differential cross sections for our values of $\sigma_{\text{exp}}$. An account of possible other singularities has been given by one of us[1] and by Shapiro.[2]

In Sec. II we give an outline of the necessary theoretical points where special attention is given to the pole term. A computer experiment is described in Sec. III.

## II. A REDUCED WIDTH AS THE RESIDUE OF A POLE

As mentioned in Sec. I we wish to propose a model-independent definition of a reduced width, which is associated with the residue of a pole in the transition amplitude. This discussion, using wave functions, is similar in spirit to the analysis of the analytic properties of stripping amplitudes using Feynman diagrams.[1] Although we restrict ourselves to the stripping reaction $A(d,p)B$, the principles are easily generalized to other nuclear reactions.

Since our main purpose is to isolate the pole term, we begin with the expression for the single-particle plane-wave Born approximation of the cross section for the

* Supported in part by the U. S. Atomic Energy Commission.
† Now at University of Washington, Seattle, Washington.
†† Now at Brandeis University, Waltham, Massachusetts.
[1] H. J. Schnitzer, Nucl. Phys. **36**, 505 (1962). This will be referred to in the text as I.
[2] I. S. Shapiro, Nucl. Phys. **28**, 244 (1961).
[3] G. Chew and F. Low, Phys. Rev. **113**, 1640 (1959).
[4] R. D. Amado, Phys. Rev. Letters **2**, 399 (1959) and Phys. Rev. **127**, 261 (1962).
[5] R. H. Bassel, R. M. Drisko, and G. R. Satchler, Oak Ridge National Laboratory Report ORNL-3240 (unpublished).

reaction $d+A \rightarrow p+B^5$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Born}} = \frac{2\mathcal{J}_B+1}{2\mathcal{J}_A+1}\frac{8(M_A+1)M_A}{(M_A+2)^2} \times\left(\frac{k_p}{k_d}\right)\left(\frac{\alpha}{1-\alpha r_t}\right)|I_l(q)|^2. \quad (3.1)$$

Here $\mathcal{J}_A$ and $\mathcal{J}_B$ are the spins of the nucleus $A$ and $B$, $M_A$ is the mass of nucleus $A$; $k_p$ and $k_d$ are proton and deuteron momenta in the c.m. system; $r_t$ is the neutron-proton triplet effective range; $i\alpha$ is the wave number of the neutron bound in the deuteron; $\mathbf{q}$ is the three-momentum transfer.

The overlap integral $I_l(q)$ is defined by

$$I_l(q)=\int_0^\infty dr\, r^2F_l(r)j_l(qr), \quad (3.2)$$

where $F_l(r)$ is the radial neutron bound-state wave function, bound in nucleus $A$, and $j_l(qr)$ is the spherical Bessel function. Expression (3.1) is, of course, not the complete expression for the stripping cross section. However, the omitted terms will not alter the residue of the pole which occurs in the complex $\cos\theta$ plane. For example, the distortions due to the rescattering of the final proton can be compared with the branch cut contributions (or singularities at $q^2=\infty$ if the rescattering occurs in a finite ranged potential) which correspond to the square diagram, Fig. 7 of reference 1. We will not exhibit the details, although a computation very similar to the one we make for the bound-state wave function is possible.[6,7]

If the interaction between the neutron and the nucleus can be described by the superposition of Yukawa potentials, then one can show that one can write[6] $F_l(r)$ as the following superposition of spherical Hankel functions:

$$rF_l(r)=-\int_0^\infty \rho_l(\sigma)\sigma r h_l^{(1)}(i\sigma r)d\sigma, \quad (3.3)$$

$$\rho_l(\sigma)=N_l\delta(\sigma-k_n)+\rho_l(\sigma)\theta(\sigma-\beta^2), \quad (3.4)$$

where $\beta^2>k_n$, $k_n$ being the neutron bound-state wave number, and $N_l$ is a constant. Then

$$F_l(r)=-k_nN_lh_l^{(1)}(ik_nr)-\int_{\beta^2}^\infty \sigma\rho_l(\sigma)h_l^{(1)}(i\sigma r)d\sigma. \quad (3.5)$$

To satisfy the boundary condition at $r=0$, we require

$$F_l(r)\underset{r\to 0}{\longrightarrow} Cr^l. \quad (3.6)$$

This implies the following relations:

$$\int_0^\infty \rho_l(\sigma)\sigma^m d\sigma=0 \quad (m=-l,\cdots,+l), \quad (3.7)$$

and

$$\int_0^\infty r^2|F|^2dr=1,$$

but otherwise $\rho_l(\sigma)$ for $\sigma>\beta^2$ is a quite arbitrary weight function. Using this wave function we have

$$I_l(q)=N_l\left(\frac{q}{k_n}\right)^l\frac{1}{q^2+k_n^2}+\int_{\beta^2}^\infty d\sigma\rho_l(\sigma)\left(\frac{q}{\sigma}\right)^l\frac{1}{q^2+\sigma^2}. \quad (3.8)$$

We see that the overlap integral separates into a pole and branch-cut contribution. The branch-cut corresponds to the diagram of Fig. 6 of I, while the pole is that of Fig. 2 of I.

It is convenient to define the dimensionless quantity $G_l$

$$G_l=[1/(2m)^{1/2}](m/k_n)_lN_l, \quad (3.9)$$

where $m$ is the nucleon mass.

The contribution of the pole term to the cross section takes the form

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{pole}} = \frac{2\mathcal{J}_B+1}{2\mathcal{J}_A+1}\frac{8M_A(M_A+1)}{(M_A+2)^2}\left(\frac{k_p}{k_d}\right) \times\frac{\alpha}{1-\alpha r_t}(2m)\left(\frac{q}{m}\right)^{2l}\frac{|G_l|^2}{(q^2+k_n^2)^2}. \quad (3.10)$$

We comment that the wave function of the neutron in the deuteron has a representation of the form (3.5). The branch cuts arising from the overlap integral of the deuteron[5] (which we have not shown here explicitly) are those of Fig. 5 of I. The Hulthén wave function is a special case of (3.5). We see in (3.9) and (3.10) that we have a connection between the residue of the stripping pole and the asymptotic normalization of the bound-state wave function. In addition, $G_l$ is an energy-independent and *model-independent* quantity which is completely analogous to the coupling constant of the field theory vertex function.[8] We wish to propose $N_l$ (or $G_l$) as a definition of the reduced width, since (in analogy with the coupling constant) this uniquely specifies the strength of a vertex function, independent of which of its "legs" is the leg of a pole diagram. Hence, one gets the same reduced width whenever a given vertex function occurs in any nuclear reaction.

We wish to emphasize that the residue of the pole, as we have defined it, does not depend on the assumption that the neutron is bound in a potential which is a superposition of Yukawa potentials. In fact, if this potential is of finite range, then $N_l$ is proportional to the usual reduced width; however, the usual definition

[6] L. Bertocchi, C. Ceolin, and M. Tonin, Nuovo Cimento **18**, 770 (1960).

[7] V. de Alfaro and M. Rossetti, Nuovo Cimento **18**, 783 (1960).

[8] R. Blankenbecler and L. F. Cook, Phys. Rev. **119**, 1745 (1960).

will depend on the potential shape if it has a Yukawa tail. For a neutron bound in a finite-ranged potential, the pole term is unaltered. For if we write $I^{(1)}$ for the overlap integral using a representation of the form of Eq. (3.3) and $I^{(2)}$ for the overlap integral for a finite ranged potential with the same $N_l$ and $k_n$ as in $I^{(1)}$, with

$$\begin{aligned} F_l(r) &= -k_n N_l h_l^{(1)}(ik_n r), \quad r > r_0, \\ &= R_l(r), \qquad\qquad\quad r \leqslant r_0, \end{aligned} \tag{3.11}$$

then $I^{(1)}-I^{(2)}$ has a branch-cut term beginning at $q^2=-\beta^2$ and a finite integral of a measurable function which can give at most singularities at $q^2=\infty$. Hence, $I^{(2)}$ has the *identical* pole term. So our pole term remains the same, independent of the choice of potential.

A simple example illustrates how a finite-ranged potential can give nearly identical results for the scattering amplitude (in the region of physical $q^2$) as a potential with a Yukawa tail, but with very different analytic properties in the $q^2$ (or $\cos\theta$) complex plane. Consider the Born approximation scattering amplitude $f_B$ obtained from the potentials

$$V^{(1)}(r) = ge^{-\mu r}/r$$

and

$$\begin{aligned} V^{(2)}(r) &= ge^{-\mu r}/r, \quad r \leqslant R, \\ &= 0, \qquad\qquad r > R. \end{aligned} \tag{3.12}$$

Now $f_B{}^{(1)}$ has a pole at $q^2=-\mu^2$, while $f_B{}^{(2)}$ has only an essential singularity at $q^2=\infty$. The exact scattering amplitude $f^{(1)}$ also has branch cuts in the $q^2$ plane, but $f^{(2)}$ has no singularities in the finite complex $q^2$ plane. However, if $R \gg 1/\mu$, the scattering amplitudes for *physical* values of $q^2$ are almost identical. However, as we have shown, this ambiguity in choice of potentials does not affect our stripping pole term, but does alter the other analytic properties of the stripping amplitude. In the next section we discuss a computer experiment whose purpose is to test the accuracy of an extrapolation to the pole to determine the reduced width.

## III. A COMPUTER EXPERIMENT

One can test the extrapolation procedure by means of a computer experiment. We have performed the extrapolation on an IBM 650 computer, using a least-squares fitting program as given by Goodwin.[9] The details of this are described in Appendix B. We used differential cross sections obtained from distorted-wave calculations where the asymptotic normalization of the bound-state wave function is known.[10] Satchler has provided us with the stripping differential cross sections for $Si^{28}(d,p)Si^{29}$ to the ground state and first excited state from the distorted-wave "Sally" program.[5] There are these three cases for each reaction.

(1) *Case* (*C*). The deuteron scatters from an optical potential of the Saxon type

$$V_d = -(80+15i)/(e^x+1)(\text{MeV}),$$
$$x = [r-1.5(28)^{1/3}]/0.6 (r \text{ in } 10^{-13} \text{ cm}).$$

The proton scatters from a potential of the form

$$V_p = -(42+8i)/(e^x+1)(\text{MeV}),$$
$$x = [r-1.3(29)^{1/3}]/0.6 \quad (r \text{ in } 10^{-13} \text{ cm}).$$

Coulomb interactions are due to a uniform charged sphere of radius $1.3A^{1/3}\times 10^{-13}$ cm.

(2) *Case* (*N*). Same as case (*C*) but with no Coulomb interaction.

(3) *Case* (*CB*). Same as case (*C*); however, the strengths of the optical potentials are changed as follows:

$$V_d = -(50+10i)/(e^x+1)(\text{MeV})$$

and

$$V_p = -(42+5i)/(e^x+1)(\text{MeV}).$$

For each of these cases the neutron is captured into an orbit with wave function taken as a harmonic oscillator wave function for $r \leqslant R = 5.3\times 10^{-13}$ cm with the correct $(n,l)$. This is matched smoothly on to a Hankel function of correct binding energy for $r > R$. Typical differential cross sections for the three cases are shown in Figs. 1 and 2.

The extrapolations were performed, as outlined in Appendix B, using Eq. (3.10) for the pole term. We did not use the bound-state wave function in the extrapolation since we wished to test the accuracy of

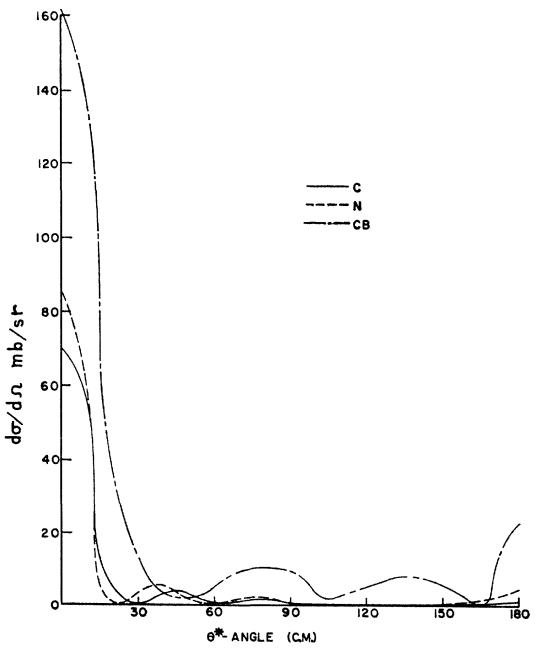


FIG. 1. Differential cross sections for computer experiment for $Si^{28}(d,p)Si^{29}$ to the ground state ($E_d$=7 MeV lab kinetic energy).

[9] L. K. Goodwin, University of California Radiation Laboratory Report UCRL-9263 (unpublished).
[10] See Appendix A.

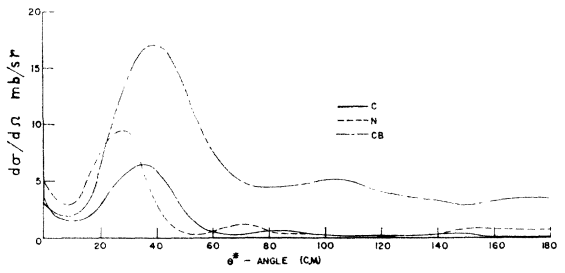


FIG. 2. Differential cross sections for computer experiment for $Si^{28}(d,p)Si^{29}$ to the first excited state ($E_d=7$ MeV lab kinetic energy).

taking the pole term alone, which should be model independent. The results for the ground state are shown in Table I and for the first excited state in Table II. These values are to be compared with the "theoretical" values $|G_0|^2=2.28$ and $|G_2|^2=930$ obtained from Appendix A. The errors are statistical and are obtained from assigning an error proportional to $1/\sqrt{\sigma}$ to the differential cross-section values. The best $M$ indicates the order of the polynomial required for the optimum extrapolation, i.e., $M=7$ means a polynomial of the form $\sum_{n=0}^{6} a_n y^n$ was required.

## IV. DISCUSSION AND CONCLUSIONS

For all cases high-order polynomials are required for the extrapolation, with typical values of $M=5$ or 6. This immediately implies that terms other than the pole are of considerable importance. These branch-cut terms not included in the extrapolation formula come in by means of the high-order polynomial required. If we had included some of these effects, e.g., the vertex function due to the bound-state neutron, then the polynomial would have been of lower order; however, in general, the extrapolation formula would not be model independent.

TABLE I. Extrapolated values of $G^2$ in the computer experiment for $Si^{28}(d,p)Si^{29}$ to the ground state.

| Case | Energy (MeV) | $G^2$ | $\delta G^2$ | Best $M$ |
|---|---|---|---|---|
| $C$ | 5 | 0.304 | 0.023 | 8 |
| | 7 | 0.614 | 0.031 | 7 |
| | 9 | 0.485 | 0.072 | 6 |
| | 11 | 0.594 | 0.065 | 6 |
| | 13 | 0.526 | 0.065 | 6 |
| | 15 | 0.372 | 0.062 | 6 |
| $N$ | 5 | 0.785 | 0.109 | 7 |
| | 7 | 0.684 | 0.089 | 7 |
| | 9 | 0.627 | 0.099 | 6 |
| | 11 | 0.604 | 0.082 | 6 |
| | 13 | 0.420 | 0.068 | 6 |
| | 15 | 0.327 | 0.066 | 6 |
| $CB$ | 5 | 0.710 | 1.52 | 6 |
| | 7 | 1.18 | 0.086 | 7 |
| | 9 | 0.600 | 0.089 | 6 |
| | 11 | 0.454 | 0.079 | 6 |
| | 13 | 0.244 | 0.075 | 6 |
| | 15 | 0 | 0.058 | 6 |
| Theoretical value $G^2=2.28$ | | | | |

TABLE II. Extrapolated values of $G^2$ in the computer experiment for $Si^{28}(d,p)Si^{29}$ to the first excited state.

| Case | Energy (MeV) | $G^2$ | $\delta G^2$ | Best $M$ |
|---|---|---|---|---|
| $C$ | 5 | 212 | 2 | 6 |
| | 7 | 228 | 13 | 7 |
| | 9 | 101 | 12 | 7 |
| | 11 | 122 | 26 | 6 |
| | 13 | 87.7 | 26 | 5 |
| | 15 | 64.9 | 19 | 6 |
| $N$ | 5 | 544 | 43 | 6 |
| | 7 | 308 | 43 | 6 |
| | 9 | 125 | 24 | 7 |
| | 11 | 111 | 29 | 6 |
| | 13 | 170 | 36 | 5 |
| | 15 | 80 | 23 | 6 |
| $CB$ | 5 | 224 | 102 | 8 |
| | 7 | 365 | 19 | 7 |
| | 9 | 597 | 46 | 5 |
| | 11 | 1190 | 71 | 5 |
| | 13 | 266 | 65 | 5 |
| | 15 | 242 | 65 | 5 |
| Theoretical value $G^2=930$ | | | | |

The values of $G^2$ should be energy independent. However, there is considerable energy dependence in the cases ($N$) and ($CB$), while the energy dependence of the extrapolated value of $G^2$ is somewhat less for the case ($C$). This suggests that the Coulomb interactions and the nuclear distortions may in some way interfere destructively. Since the value of $G^2$ found does depend on energy, the pole term is not sufficiently isolated from the branch cuts to allow the approximation of Eq. (B5), and, hence, the $G^2$ we found includes other contributions besides the residue of the pole.

The absolute values for $G^2$ found by extrapolation are typically of the order of a factor 4 or so too small from the value expected theoretically. This also supports the conclusions of the previous two paragraphs. However, on the optimistic side, the values found for $G^2$ varied less from case to case [e.g., case ($C$) compared with case ($CB$)] than did the differential cross sections. If the pole was completely isolated by our procedure, then $G^2$ would be the same for the three cases since the neutron bound-state wave function is the same. The procedure of Appendix B was clearly more successful than a "naive" extrapolation. This is shown by considering the pole term, Eq. (3.10), for the first excited state, which is an $l=2$ capture. In that case $(d\sigma/d\Omega)_{\text{pole}}$ increases monotonically in the backward direction, which is very different from the differential cross sections. However, the values found for $G^2$ are not too different from the expected values.

These results indicate that the extrapolation to the pole using Eq. (3.10) is capable of giving a model-independent value for a reduced width which is accurate to better than an order of magnitude. One should expect even more accurate results if the bound-state

wave function is included in the extrapolation.[11] However, this will in general introduce additional free parameters and will also depend on a specific model.


## ACKNOWLEDGMENTS

We are deeply indebted to Dr. G. R. Satchler for providing us with the cross sections. We wish to thank Professor N. Austern, Professor J. B. French, and Professor M. MacFarlane for useful discussions.


## APPENDIX A

We will find values for the asymptotic normalization constants of the captured neutron by means of the following prescription[5] In the stripping reaction $Si^{28}(d,p)Si^{29}$ the neutron is captured in a shell corresponding to the appropriate harmonic oscillator level. We approximate the neutron wave function for $r\leqslant R_n$ by the proper harmonic oscillator function and match this smoothly to a spherical Hankel function with imaginary argument at $r=R_n$. Given the binding energy of the neutron, we will then be able to find values of the oscillator strength, so that the approximate shape of the wave function is known. Normalization of this wave function to unity then given us the necessary information.

For the ground state of $Si^{29}$ we have $l=0$ and the radial wave function will be for $r\leqslant R_n$ (and principle quantum number $n=3$)

$$R_0(r)=\alpha_0 D_0 e^{-\frac{1}{2}\alpha_0^2 r^2}(\tfrac{3}{2}-\alpha_0^2 r^2), \tag{A1}$$

and for the first excited state ($l=2$)

$$R_2(r)=\alpha_2 D_2(\alpha_2 r)^2 e^{-\frac{1}{2}\alpha_2^2 r^2}, \tag{A2}$$

$\alpha$ being the oscillator strength which should have comparable values for the two cases. The radius $R_n$ will be taken the same for both cases and $D_{0,2}$ are constants. The functions $rR_0$ and $rR_2$ and their derivatives will be matched to $N_0(k_{n_0}r)h_0^{(1)}(ik_{n_0}r)$ and $N_2(k_{n_2}r)h_2^{(1)}(ik_{n_2}r)$, respectively; $ik_{n_0}$ and $ik_{n_2}$ are the wave numbers corresponding to the two binding energies of the neutron; and $N_0$, $N_2$ are constants. Introduce

$$\zeta_0=\alpha_0 R_n, \quad \lambda_0=k_{n_0}R_n, \quad \zeta_2=\alpha_2 R_n, \quad \lambda_2=k_{n_2}R_n.$$

We then have for the ground state

$$\zeta_0^2=(2\lambda_0+9)/4\pm\tfrac{1}{4}(4\lambda_0^2+12\lambda_0+57)^{1/2}, \tag{A3}$$

and the first excited state

$$\zeta_2^2=2+(\lambda_2^3+4\lambda_2^2+9\lambda_2+9)/(\lambda_2^2+3\lambda_2+3). \tag{A4}$$

The condition $\int dr\, r^2|R(r)|^2=1$ will give for the ground state

$$\frac{|N_0|^2}{2k_{n_0}}e^{-2\lambda}+\frac{|D_0|^2}{4\alpha_0}[(-2\zeta_0^5+\zeta_0^3-3\zeta_0)e^{-\zeta_0^2}+\tfrac{3}{2}\pi^{1/2}\,\mathrm{erf}\zeta_0]=1, \tag{A5}$$

and for the $l=2$ case

$$\frac{|D_2|^2}{\alpha_2}\left[\left(-\frac{\zeta_2^5}{2}-\frac{5\zeta_2^3}{4}-\frac{15}{8}\zeta_2\right)e^{-\zeta_2^2}+\tfrac{15}{16}\pi^{1/2}\,\mathrm{erf}\zeta_2\right]+\frac{|N_2|^2}{k_n}\left[\frac{1}{2}+\frac{3}{\lambda_2}+\frac{6}{\lambda_2^2}+\frac{3}{\lambda_2^3}\right]e^{-2\lambda_2}=1. \tag{A6}$$

Given $\lambda_0$ and $\lambda_2$ one can now solve for $\zeta_0$ and $\zeta_2$ and the requirement that $\zeta_0$ and $\zeta_2$ should be almost equal excludes one of the possibilities in (A3).

## APPENDIX B. EXTRAPOLATION PROCEDURE

Here we review the procedure for extrapolating to a pole in order to present our calculation in as explicite form as possible. Consider, for example, a scattering amplitude which can be written, for fixed total energy, as

$$F(s,\cos\theta)=\frac{gG}{A-B\cos\theta}+\int_{(\cos\theta)_0}^{\infty}dx'\frac{\rho(x')}{(x'-\cos\theta)}, \tag{B1}$$

where

$$(\cos\theta)_0>A/B, \quad |F|^2=d\sigma/d\Omega,$$

$A$, $B$, $g$, $G$ are constants. Then it is clear that

$$\lim_{\cos\theta\to A/B}(A-B\cos\theta)^2 d\sigma/d\Omega=gG. \tag{B2}$$

Define

$$E(s,\cos\theta)\equiv(A-B\cos\theta)^2 d\sigma/d\Omega. \tag{B3}$$

We refer to $E(s,\cos\theta)$ as the extrapolation function at fixed total energy $S$. Now, with the representation (B1), the following expansion is possible

$$E(s,\cos\theta)=\sum_{n=0}^{\infty}b_n(A-B\cos\theta)^n. \tag{B4}$$

In extrapolations it is assumed that one may write

$$E(s,\cos\theta)\simeq\sum_{n=0}^{M}(A-B\cos\theta)^n a_n. \tag{B5}$$

Then the best value for $M$ is selected by statistical methods.[12] We can summarize the procedure of Cziffra and Moravcsik as follows:

(1) We consider the extrapolation for a single energy only, thus we can suppress the parameter $s$. One computes the data points $E(\cos\theta)$ with accompanying errors from the experimental data.

(2) Fit a least-squares polynomial of form (B5) to $E(\cos\theta)$ for fixed $M$. Compute the value of $\chi_M^2=\sum_i \omega_i\epsilon_i^2$ (the weighted least-square error) for this value of $M$.

(3) Plot $\chi_M^2$ vs $M$. One should find the value of $\chi_M^2$ to decrease rapidly with increasing $M$ and then reach a plateau which will increase slowly for further increases in $M$. The best value of $M$ is at the minimum in $\chi_M^2$ just before the beginning of the plateau.

(4) In ambiguous cases one may apply the Fisher $F$

[11] This has been emphasized by R. Amado (private communication).

[12] This is discussed in detail by P. Cziffra and M. Moravcsik, University of California Radiation Laboratory Report UCRL-8523 (rev.) (unpublished).

test, which compares $\chi_m^2$ with $\chi_{m+1}^2$ to find the best value of $M$.[12]

(5) For the best value of $M$, one uses the least-square curve (B5) to compute

$$E(A/B)=gG=a_0.$$

We have described the extrapolation procedure for cross sections for which (B1) is valid, however the procedure may be applied to any cross section in which there is an isolated pole in $(\cos\theta)$, and for which (B5) is a valid approximation. In our work, we have assumed this to be true.

PHYSICAL REVIEW VOLUME 129, NUMBER 2 15 JANUARY 1963

# Lifetimes of the Low-Energy $M1$ Transitions in $La^{137}$ and $Kr^{83}$

S. L. RUBY*, Y. HAZONI, AND M. PASTERNAK
*Soreq Research Establishment, Atomic Energy Commission, Rehovoth, Israel*
(Received 16 August 1962)

Using conventional time to pulse-height techniques, the lifetimes of the low-lying levels in $La^{137}$ and $Kr^{83}$ have been measured, despite the low $\gamma$-ray energies (10 and 9.3 keV) and the large internal conversion coefficients (130 and 11). The results of $89\pm4$ and $147\pm4$ nsec, respectively, correspond to retardation factors of 370 and 45 over the single-particle model predictions for those $M1$ transitions.

THE measurement of the lifetime of a nuclear state is useful either as a method of clarifying a tangled decay scheme or, if the decay scheme is clear, as a means of evaluating nuclear matrix elements. In the two cases considered here, the pertinent parts of the decay scheme have been established by earlier measurements.[1,2] Therefore, the lifetimes, when combined with values for the internal conversion coefficients, can be used to find the retardation factors compared to the single-particle predictions for these decays. These are essentially equal to the nuclear matrix element of the known operator (magnetic dipole) between the ground and first excited states of these nuclei. Such information is of definite value for the testing of nuclear models. The two found here are to be added to only ten previously measured $M1$ transitions for $A>20$.

A second reason for being interested in the lifetimes of low-lying states is that long lifetimes and low energy offer the possibility of using the nucleus for recoil-free nuclear resonance experiments. The recent history of the exploitation of $Fe^{57}$ by means of the Mössbauer Effect[3] is clear testimony of the possible value of such nuclei.

## EXPERIMENTAL TECHNIQUES

The main experimental problems were created by the large internal conversion coefficients and the low energy of the transitions to be explored. In the absence of a low-energy magnetic spectrometer, it was decided to attempt the measurement by a straightforward extension to lower energies of the time-to-pulse-height delayed coincidence techniques now in common use,[4] using scintillation counters as detectors. A modification of a circuit of Simms[5] was constructed for the time-to-pulse-height conversion, and 6810-A phototubes feeding into a limiter and shorted delay line were used to establish the standard pulses. A slow channel for pulse-height selection was set up for each detector—only if each pulse fell within selected values was the multichannel analyzer allowed to measure the output of the fast time-to-height converter.

Examination of the decay scheme shown in Fig. 1 shows that the distribution of time delays between the detection of a 29-keV $K$ x ray of $La^{137}$ and the detection of the 10-keV $\gamma$ ray contains the desired information. In the case of $Kr^{83}$, as seen in Fig. 2, the energies are even lower; here the $K$ x ray is only 12.6 keV and the $\gamma$ ray 9.3 keV. The experimental question was whether suitable energy and time resolution was available.

Owing largely to the excellent photocathode response and low dark current emission of the 6810 A's, little difficulty was found in either regard. Using an anthracene crystal mounted directly on the phototube as a scintillator, it was found that one photoelectron was emitted for every 1000 eV dissipated in the phosphor. In addition, it was found that under dark conditions, there were less than thirty pulses per second larger than two photoelectrons. This number results from the "pile-up" of single photoelectrons emitted, mainly thermionically, at the photocathode. In the presence of a radioactive source, the number of single photoelectrons goes up, owing mainly to light emission from

* On leave from Research Laboratories, Westinghouse Electric Corporation, Pittsburgh 35, Pennsylvania.

[1] A. R. Brosi and B. K. Ketelle, Phys. Rev. **103**, 917 (1956), see p. 920.

[2] P. T. Barrett, Proc. Phys. Soc. (London) **65A**, 450 (1952).

[3] H. Frauenfelder, *Mössbauer Effect* (W. A. Benjamin, Inc., New York, 1962).

[4] R. E. Bell, in *Beta- and Gamma-Ray Spectroscopy*, edited by K. Seigbahn (North-Holland Publishing Company, Amsterdam, 1955).

[5] P. C. Simms, Rev. Sci. Instr. **32**, 895 (1961).